3. Parametric estimation and limit theorems

Task 1. */2 points/* Generate random sample $X_1, X_2, \ldots, X_n$ from a distribution (see table) with mean $E[X_i] = \mu$ and variance $Var[X_i] = \sigma^2$. Calculate sample mean $\overline{X}_n$ and sample variance S_n^2:

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \tag{1}$$

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2. \tag{2}$$

Experimentally observe the bias of each estimator as $n \to \infty$. Plot the estimator value (1) or (2) in function of sample size n. Draw a horizontal line on the same plot for true value of mean μ or variance σ^2 respectively. Are sample mean $\overline{X}_n$ and sample variance S_n^2 unbiased?

No	distribution	p.d.f.	μ	σ^2
1	normal $\mathcal{N}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$	μ	σ^2
2	uniform $\mathcal{U}[a, b]$	$f(x) = \frac{1}{b-a}\mathbf{1}\{a \leqslant x \leqslant b\}$	$\mu = \frac{1}{2}(a+b)$	$\sigma^2 = \frac{1}{12}(b-a)^2$
3	$Beta(\alpha, \beta)$	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}\{0 \leqslant x \leqslant 1\}$	$\mu = \frac{\alpha}{\alpha+\beta}$	$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
4	exponential $Exp(\lambda)$	$f(x) = \lambda \exp(-\lambda x)\,\mathbf{1}\{x \geqslant 0\}$	$\mu = \frac{1}{\lambda}$	$\sigma^2 = \frac{1}{\lambda^2}$

Task 2. */3 points/* Experimentally illustrate Central Limit Theorem:

Lindeberg-Lévy CLT. *Suppose $X_1, X_2, \ldots X_n$ is a sequence of i.i.d. random variables with $E[X_i] = \mu$ and $Var[X_i] = \sigma^2 < \infty$. Then* as $n \to \infty$

$$\frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{n} X_i - n\mu\right) \xrightarrow{D} \mathcal{N}(0,1). \tag{3}$$

Method:

1. Generate random sample $X_1, X_2, \ldots X_n$ of size n. Calculate $Y_1 = \frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{n} X_i - n\mu\right)$.
2. Generate second random sample $X_1, X_2, \ldots X_n$ of size n. Calculate $Y_2 = \frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^{n} X_i - n\mu\right)$.
3. ... (repeat it k times)
4. Draw histogram for random variables $Y_1, Y_2, \ldots Y_k$ and compare its shape with theoretical pdf of $\mathcal{N}(0,1)$.
5. Repeat steps 1-4 for larger sample size n.
6. Repeat steps 1-5 for another distribution of random sample $X_1, X_2, \ldots X_n$ (see table from task 1).

/Total: 5 points/